UDC 539.3

ON A PLANE PROBLEM OF ELASTICITY THEORY FOR A HALF-STRIP

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In connection with the construction of a refined theory of plates and shells by an asymptotic method [1— 6], the question of the plane problem of elasticity theory for a half-strip arises. In the asymptotic approach, the plate (shell) region near the edge (or other perturbation lines) is singled out for special consideration. The state of stress of the boundary layer, which damps out rapidly with distance from the edge, must be taken into account to determine the state of stress in this region. The construction of the boundary layer [4 — 6] reduces to an iteration process in each of whose steps the plane and antiplane problems of elasticity theory for a half-strip should be solved. In the first step it is hence necessary to solve homogeneous equations with homogeneous boundary conditions for the stresses on the longitudinal boundaries and with different conditions at the edge. At subsequent stages the solutions of the plane and antiplane problems of elasticity theory for a half-strip for given volume forces and for given stresses on the longitudinal boundaries are required.

The question of the necessary and sufficient conditions for the existence of damped solutions of the plane problem of elasticity theory for a half-strip with free longitudinal boundaries in the absence of volume forces is examined in [7].

An extension of the results in [7] to the case when volume forces act on the half-strip and stresses are given on the longitudinal boundaries is given below.

1. Let us turn to the solution of the above-mentioned problem for a half-strip. We agree to consider quantities with the dimension of a length to be referred to the half-thickness h of a half-strip, and the volume forces to be referred to h^{-1}.

Let us consider the half-strip $0 \leqslant x < \infty$, $-1 \leqslant y \leqslant 1$ with the following conditions specified on the edge $x = 0$:

$$\sigma_x(0, y) = f_1(y), \quad \sigma_{xy}(0, y) = f_2(y) \quad \text{(problem 1)} \tag{1.1}$$

$$2\mu u(0, y) = f_1(y), \quad \sigma_{xy}(0, y) = f_2(y) \quad \text{(problem 2)} \tag{1.2}$$

$$\sigma_x(0, y) = f_1(y), \quad 2\mu v(0, y) = f_2(y) \quad \text{(problem 3)} \tag{1.3}$$

$$2\mu u(0, y) = f_1(y), \quad 2\mu v(0, y) = f_2(y) \quad \text{(problem 4)} \tag{1.4}$$

As in [7], we examine the skew-symmetric and symmetric strains of the half-strip separately. The boundary conditions on the longitudinal boundaries for all four problems have the form

$$\sigma_x(x, 1) = g_1(x), \quad \sigma_{xy}(x, 1) = g_2(x) \tag{1.5}$$

Let us set up necessary and sufficient conditions imposed on the boundary functions $f_1(y)$ and $f_2(y)$, upon compliance with which the solution of the inhomogeneous Lamé equations

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x\, \partial y} = -r_x(x, y) \tag{1.6}$$

$$\mu \frac{\partial^2 v}{\partial x^2} + (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x\, \partial y} = -r_y(x, y)$$

satisfying the given conditions (1.5) on the longitudinal boundaries will be damped out in the direction of the x-axis.

We construct the damping solution of the problem for the boundary functions corresponding to the damping conditions. We assume with respect to the functions in the right sides of (1.5) and (1.6) that they damp out exponentially in the direction of the x-axis. We shall examine the solution of (1.6) in the class of damping functions.

Let us use a Laplace transformation in the variable x to construct the solution to (1.6). We set

$$U(p, y) = \int_0^\infty u(x, y)\, e^{-px}\, dx, \quad V(p, y) = \int_0^\infty v(x, y)\, e^{-px}\, dx \tag{1.7}$$

and we obtain an inhomogeneous system of ordinary equations for $U(p, y)$ and $V(p, y)$:

$$(\lambda + 2\mu)\, p^2 U + \mu \frac{\partial^2 U}{\partial y^2} + (\lambda + \mu)\, p \frac{\partial V}{\partial y} = \Phi(p, y) \tag{1.8}$$

$$\mu p^2 V + (\lambda + 2\mu)\frac{\partial^2 V}{\partial y^2} + (\lambda + \mu)\, p \frac{\partial U}{\partial y} = \Psi(p, y)$$

$$\Phi(p, y) = -R_x(p, y) + \sigma_x(0, y) + \mu \left.\frac{\partial v}{\partial y}\right|_{x=0} + (\lambda + 2\mu)\, pu(0, y)$$

$$\Psi(p, y) = -R_y(p, y) + \sigma_{xy}(0, y) + \lambda \left.\frac{\partial u}{\partial y}\right|_{x=0} + \mu pv(0, y)$$

Here $R_x(p, y)$, $R_y(p, y)$ are the transforms of $r_x(0, y)$, $r_y(0, y)$.

The general solution of (1.8) is

$$\begin{aligned} U(p, y) &= a_1(p)\sin py + a_2(p)\cos py + \\ &\quad a_3(p)\, py\cos py + a_4(p)\, py \sin py + U_0(p, y) \\ V(p, y) &= (-a_2(p) - \varkappa_1 a_4(p))\sin py + \\ &\quad (a_1(p) - \varkappa_1 a_3(p))\cos py + a_4(p)\, py\cos py - \\ &\quad a_3(p)\, py\sin py + V_0(p, y) \end{aligned} \tag{1.9}$$

where $U_0(p, y)$, $V_0(p, y)$ is a particular solution of (1.9). Here

$$\begin{aligned} U_0(p, y) &= b_1(p, y)\sin py + b_2(p, y)\cos py + \\ &\quad b_3(p, y)\, py\cos py + b_4(p, y)\, py\sin py \\ V_0(p, y) &= (-b_2(p, y) - \varkappa_1 b_4(p, y))\sin py + (b_1(p, y) - \\ &\quad \varkappa_1 b_3(p, y))\cos py + b_4(p, y)\, py\cos py - b_3(p, y)\, py \sin py \end{aligned} \tag{1.10}$$

$$\begin{aligned} b_1(p, y) &= \frac{\varkappa}{p}\int_{y_1}^{y} [\Psi(p, y)\, py\cos py + \\ &\quad \Phi(p, y)(\varkappa_1 \cos py - py\sin py)]\, dy \\ b_2(p, y) &= \frac{\varkappa}{p}\int_{y_2}^{y} [-\Psi(p, y)\, py\sin py - \\ &\quad \Phi(p, y)(\varkappa_1 \sin py + py\cos py)]\, dy \end{aligned} \tag{1.11}$$

$$b_3(p, y) = \frac{\varkappa}{p}\int_{y_1}^{y} [\Psi(p, y)\sin py + \Phi(p, y)\cos py]\, dy$$

$$b_4(p, y) = \frac{\varkappa}{p}\int_{y_2}^{y} [-\Psi(p, y)\cos py + \Phi(p, y)\sin py]\, dy$$

$$\varkappa_1 = \frac{\lambda + 3\mu}{\lambda + \mu}, \quad \varkappa = \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)}$$

We express the stresses in conditions (1.5) in terms of the displacements and apply a Laplace transform. We obtain

$$-\lambda u(0,1) + \lambda p U(p, 1) + (\lambda + 2\mu)\frac{\partial V}{\partial y}\Big|_{y=1} = G_1(p) \qquad (1.12)$$

$$-\mu v(0,1) + \mu p V(p, 1) + \mu\frac{\partial U}{\partial y}\Big|_{y=1} = G_2(p)$$

Here $G_1(p)$, $G_2(p)$ are transforms of the functions $g_1(x)$ and $g_2(x)$.

2. Let us first consider skew-symmetric strain of the half-strip. Then $a_2(p)$, $a_4(p)$ in (1.9) equal zero, and $y_1 = -1$ and $y_2 = 0$ in (1.11). Determining $a_1(p)$, $a_3(p)$ from (1.12), we obtain

$$a_1(p) = \frac{1}{p\varphi(p)}\Big\{\frac{1}{2\mu}(G_1(p) + \lambda u(0,1))\Big(\frac{\mu}{\lambda+\mu}\cos p + p\sin p\Big) + \qquad (2.1)$$

$$\frac{1}{2\mu}(G_2(p) + \lambda v(0,1))\Big(\frac{\lambda + 2\mu}{\lambda+\mu}\sin p - p\cos p\Big) +$$

$$pb_2(p, 1)\Big(\frac{\mu}{\lambda+\mu} + \sin^2 p\Big) + pb_4(p, 1)\Big(\frac{\mu(\lambda+2\mu)}{(\lambda+\mu)^2} + p^2\Big)\Big\}$$

$$a_3(p) = \frac{1}{p\varphi(p)}\Big\{\frac{1}{2\mu}(G_1(p) + \lambda u(0,1))\cos p + \frac{1}{2\mu}(G_2(p) + \lambda v(0,1)) \times$$

$$\sin p + pb_2(p, 1) + pb_4(p, 1)\Big(\frac{\mu}{\lambda+\mu} + \cos^2 p\Big)\Big\}$$

$$\varphi(p) = \sin p\cos p - p$$

The transforms $G_1(p)$, $G_2(p)$, $R_x(p, y)$, $R_y(p, y)$ of the given functions and the quantities $u(0, y)$, $v(0, y)$, $\sigma_x(0, y)$, $\sigma_{xy}(0, y)$ enter into the expressions for $U(p, y)$ and $V(p, y)$. Among these latter, two quantities are known for each of the four problems (1.1)—(1.4) from the boundary conditions on the butt, while the other two are unknown. The question of their determination will be examined later.

The functions $U(p, y)$, $V(p, y)$ can have singularities in the roots of the equation $\varphi(p) = 0$ in the complex p plane and at the singular points $G_1(p)$, $G_2(p)$, $R_x(p, y)$, $R_y(p, y)$. Since the functions $g_1(x)$, $g_2(x)$, $r_x(x, y)$, $r_y(x, y)$ damp out exponentially in the x direction, then their transforms have no singularities at the origin and in the right half-plane.

Equation $\varphi(p)=0$ has a third order root at the origin $p = 0$ and an infinite set of four first order complex roots p_n, $\bar{p}_n$, $-\bar{p}_n$, $-p_n$ $(n = 1, 2, \ldots)$.

Since $u(x, y)$, $v(x, y)$ are in the class of damped functions, then their transforms should have no singularities in the right half-plane including the imaginary axis. For this it is necessary and sufficient that the residues $U(p, y)e^{px}$, $V(p, y)e^{px}$ vanish relative to the poles p_n, $\bar{p}_n$ with positive real part, and relative to the pole $p = 0$

The residues $U\,(p,\,x)\,e^{px}$, $V\,(p,\,x)e^{px}$ relative to the pole $p = p_n$ are

$$\operatorname{res}_{p_n} U\,(p,\,y)\,e^{px} = \frac{\varkappa}{2p_n \sin^2 p_n}\,(F\,(p_n) + H\,(p_n))\,h\,(p_n,\,y)\,e^{p_n x} \qquad (2.2)$$

$$\operatorname{res}_{p_n} V\,(p,\,y)\,e^{px} = \frac{\varkappa}{2p_n \sin^2 p_n}\,(F\,(p_n) + H\,(p_n))\,g\,(p_n,\,y)\,e^{p_n x}$$

where

$$F\,(p_n) = \int_0^1 [\sigma_x(0,\,y)\,h\,(p_n,\,y) + \sigma_{xy}(0,\,y)\,g\,(p_n,\,y) + 2\mu u\,(0,\,y)\,s\,(p_n,\,y) + 2\mu v\,(0,\,y)\,t\,(p_n,\,y)]\,dy \qquad (2.3)$$

$$H\,(p_n) = -\frac{\lambda+2\mu}{\lambda+\mu}\left[\cos p_n \int_0^\infty g_1(x)\,e^{-p_n x}\,dx + \sin p_n \int_0^\infty g_2(x)\,e^{-p_n x}\,dx\right] - \int_0^1 \left[h\,(p_n,\,y)\int_0^\infty r_x(x,\,y)\,e^{-p_n x}\,dx + g\,(p_n,\,y)\int_0^\infty r_y(x,\,y)\,e^{-p_n x}\,dx\right] dy$$

$$h\,(p,\,y) = py\cos py + \left(\frac{\lambda+2\mu}{\lambda+\mu} - \cos^2 p\right)\sin py$$

$$g\,(p,\,y) = py\sin py + \left(\frac{\mu}{\lambda+\mu} + \cos^2 p\right)\cos py$$

$$s\,(p,\,y) = p\,(py\cos py + (1 + \sin^2 p)\sin py)$$

$$t\,(p,\,y) = p\,(py\sin py - \sin^2 p\cos py)$$

The residues $U\,(p,\,y)\,y^{px}$, $V\,(p,\,y)\,e^{px}$ relative to the pole $p = 0$ are (*)

$$\operatorname{res}_{p=0} Ue^{px} = -\left[\left(3x^2 + \frac{6}{5} - \frac{3\lambda+8\mu}{\lambda+2\mu}\,y^2\right)A_1 + 2xA_2 + A_3\right]y \qquad (2.4)$$

$$\operatorname{res}_{p=0} Ve^{px} = \left[x^2 - \frac{6\,(9\lambda+8\mu)}{5\,(\lambda+2\mu)} + \frac{3\lambda}{5\,(\lambda+2\mu)}\,y^2\right)xA_1 + \left(x^2 + \frac{2}{5} + \frac{\lambda}{\lambda+2\mu}\,y^2\right)A_2 + A_3 x + A_4$$

$$A_1 = -\frac{\lambda+2\mu}{8\mu\,(\lambda+\mu)}\left(\int_0^1 \sigma_{xy}(0,\,y)\,dy - \alpha\right)$$

$$A_2 = -\frac{3\,(\lambda+2\mu)}{8\mu\,(\lambda+\mu)}\left(\int_0^1 y\sigma_x(0,\,y)\,dy - \beta\right)$$

$$A_3' = -\frac{3}{2\mu}\left(\frac{\lambda}{4\,(\lambda+\mu)}\int_0^1 y^2\sigma_{xy}(0,\,y)\,dy + 2\mu\int_0^1 yu\,(0,\,y)\,dy - \gamma\right)$$

$$A_4 \quad \frac{3}{4\mu}\left(\frac{3\lambda+4\mu}{6\,(\lambda+\mu)}\int_0^1 y^3\sigma_x(0,\,y)\,dy + 2\mu\int_0^1 (1-y^2)\,v\,(0,\,y)\,dy - \delta\right)$$

The quantities α, β, γ, δ are defined in terms of the given values $g_1(x)$, $g_2(x)$,

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*) A mistake in the expressions for the residues $U\,(p,\,y)e^{px}$, $V\,(p,\,y)e^{px}$ relative to the pole $p = 0$ is made in [7]: the coefficients for A_1, A_2 were not written down completely, which did not affect the subsequent results.

$r_x(x, y)$, $r_y(x, y)$ by the formulas

$$\alpha = \int_0^\infty \left(g_1(x) + \int_0^1 r_y(x, y)\, dy\right) dx \tag{2.5}$$

$$\beta = \int_0^\infty \left[- x g_1(x) + g_2(x) + \int_0^1 (y r_x(x, y) - x r_y(x, y))\, dy\right] dx$$

$$\gamma = \frac{\lambda + 2\mu}{4(\lambda + \mu)} \int_0^\infty \left\{\left(\frac{\lambda}{\lambda + 2\mu} + x^2\right) g_1(x) - 2x g_2(x) + \right.$$

$$\left. \int_0^1 \left[- 2xy r_x(x, y) + \left(x^2 + \frac{\lambda}{\lambda + 2\mu} y^2\right) r_y(x, y)\right] dy\right\} dx$$

$$\delta = \frac{\lambda + 2\mu}{2(\lambda + \mu)} \int_0^\infty \left\{\left(- \frac{3\lambda + 4\mu}{\lambda + 2\mu} + \frac{1}{3} x^2\right) x g_1(x, y) + \right.$$

$$\left(\frac{3\lambda + 4\mu}{3(\lambda + 2\mu)} - x^2\right) g_2(x, y) + \int_0^1 \left[\left(- x^2 + \frac{3\lambda + 4\mu}{3(\lambda + 2\mu)} y^2\right) y r_x(x, y) + \right.$$

$$\left. \left. \left(- \frac{4(\lambda + \mu)}{\lambda + 2\mu} + \frac{1}{3} x^2 + \frac{\lambda}{\lambda + 2\mu} y^2\right) x r_y(x, y)\right] dy\right\} dx$$

It follows from (2.2), (2.4) that for the residues $U(p, y)\, e^{px}$, $V(p, y)\, e^{px}$ relative to the poles with positive real part and relative to the pole $p = 0$ to vanish, it is necessary and sufficient to comply with the conditions

$$F(p_n) + H(p_n) = 0 \quad (n = 1, 2, \ldots) \tag{2.6}$$

$$A_i = 0 \quad (i = 1, 2, 3, 4) \tag{2.7}$$

The system of conditions (2.7) can be represented in the form

$$\int_0^1 \sigma_{xy}(0, y)\, dy = \alpha \tag{2.8}$$

$$\int_0^1 y \sigma_x(0, y)\, dy = \beta \tag{2.9}$$

$$\frac{\lambda}{4(\lambda + \mu)} \int_0^1 y^2 \sigma_{xy}(0, y)\, dy + 2\mu \int_0^1 y u(0, y)\, dy = \gamma \tag{2.10}$$

$$\frac{3\lambda + 4\mu}{6(\lambda + \mu)} \int_0^1 y^3 \sigma_x(0, y)\, dy + 2\mu \int_0^1 (1 - y^2)\, v(0, y)\, dy = \delta \tag{2.11}$$

For each of the four problems (1.1) — (1.4), the system of conditions (2.6), (2.8) — (2.11) permits determination of those of the quantities $u(0, y)$, $v(0, y)$, $\sigma_x(0, y)$, $\sigma_{xy}(0, y)$ which are unknown from the boundary conditions at $x = 0$, and obtaining two conditions imposed on the boundary functions which are necessary and sufficient for the existence of the damped solution.

To pass to the transforms $U(p, y)$, $V(p, y)$ in (2.1) to the originals $u(x, y)$, $v(x, y)$, let us represent $U(p, y)$, $V(p, y)$ of the sum of two components

$$U(p, y) = U^{(1)}(p, y) + U^{(2)}(p, y), V(p, y) = V^{(1)}(p, y) + V^{2}(p, y) \quad (2.12)$$

The components $U^{(1)}(p, y)$, $V^{(1)}(p, y)$ correspond to transforms for the displacements in the half-strip in the absence of volume forces and stresses on the longitudinal boundaries. Transforms of the volume forces and stresses at $y = \pm 1$ enter into the terms $U^{(2)}(p, y)$, $V^{(2)}(p, y)$.

We use the inversion theorem to find the originals for $U^{(1)}(p, y)$, $V^{(1)}(p, y)$ and the convolution theorem, the inversion theorem, and the theorem on integration with respect to a parameter for $U^{(2)}(p, y)$, $V^{(2)}(p, y)$. We consequently obtain

$$u(x, y) = \sum_{n=1}^{\infty} (u^*(-p_n, x, y) + u^*(-\bar{p}_n, x, y)) \quad (2.13)$$

$$v(x, y) = \sum_{n=1}^{\infty} (v^*(-p_n, x, y) + v^*(-\bar{p}_n, x, y))$$

$$u^*(-p_n, x, y) = \frac{\lambda + \mu}{4\mu(\lambda + 2\mu)} \frac{1}{p_n \sin^2 p_n} [F(-p_n) e^{-p_n x} + K(-p_n, x)] h(p_n, y)$$

$$v^*(-p_n, x, y) = \frac{\lambda + \mu}{4\mu(\lambda + 2\mu)} \frac{1}{p_n \sin^2 p_n} [F(-p_n) e^{-p_n x} + K(-p_n, x)] g(p_n, y)$$

The values of $F(-p_n)$ are given in (2.3) and for $K(-p_n, x)$ we have

$$K(-p_n, x) = \int_0^x \left\{ \frac{\lambda + 2\mu}{\lambda + \mu} (-\cos p_n g_1(\xi) + \sin p_n g_2(\xi)) - \int_0^1 [-h(p_n, y) r_x(\xi, y) + g(p_n, y) r_y(\xi, y)] \, dy \right\} e^{-p_n(x-\xi)} d\xi \quad (2.14)$$

The expressions (2.13) for the displacments still contain unknown quantities which enter into $F(-p_n)$ and $F(-\bar{p}_n)$. We examine the question of determining them and of setting up the damping conditions for each of the problems (1.1) — (1.4).

We have two damping conditions for problem 1 from (1.1), (2.8) and (2.9):

$$\int_0^1 f_2(y) \, dy = \alpha, \quad \int_0^1 f_1(y) y \, dy = \beta \quad (2.15)$$

The values of α, β are given in (2.5). The conditions obtained are static in nature. They can be obtained from the equilibrium conditions for a half-strip subjected to all the effects applied to it.

The system of conditions (2.10), (2.11) and (2.6) is used to determine the unknowns $u(0, y)$, $v(0, y)$ in the solution (2.13). The method of determining these quantities from the system of conditions mentioned is given in [7].

We have two damping conditions for problem 2 from (1.2) and (2.8), (2.10)

$$\int_0^1 f_2(y) \, dy = \alpha, \quad \frac{\lambda}{4(\lambda + \mu)} \int_0^1 f_2(y) y^2 \, dy + \int_0^1 f_1(y) y \, dy = \gamma \quad (2.16)$$

Only the first of these conditions can be obtained from the static equilibrium conditions for a half-strip.

The system of conditions (2.9), (2.11) and (2.6) permits determination of the unknowns $\sigma_x(0, y)$, $v(0, y)$ However, their determination can be avoided in constructing the damped solution of the problem in the case under consideration since the system of conditions (2.6) permits elimination of $\sigma_x(0, y)$, $v(0, y)$ from the solution (2.13). In fact, we have from (2.3)

$$F(-p_n) = \int_0^1 [-\sigma_x(0, y)\, h(p_n, y) + \sigma_{xy}(0, y)\, g(p_n, y) + 2\mu u(0, y)\, s(p_n, y) - 2\mu v(0, y)\, t(p_n, y)]\, dy \qquad (2.17)$$

Adding the quantity $F(p_n) + H(p_n)$, which equals zero according to (2.6), to the right side of (2.17), we obtain an expression for $F(-p_n)$ in terms of quantities known from the conditions (1.2) at $x = 0$

$$F(-p_n) = 2\int_0^1 [f_2(y)\, g(p_n, y) + f_1(y)\, s(p_n, y)]\, dy + H(p_n) \qquad (2.18)$$

We have two damping conditions for problem 3 from (1.3) and (2.9), (2.11)

$$\int_0^1 f_1(y)\, y\, dy = \beta, \quad \frac{3\lambda + 4\mu}{6(\lambda + \mu)} \int_0^1 f_1(y)\, y^3\, dy + \int_0^1 (1 - y^2)\, f_2(y)\, dy = \delta \qquad (2.19)$$

Only one of these is static in nature. The system of conditions (2.6) permits elimination of the unknowns $u(0, y)$, $\sigma_{xy}(0, y)$ from the solution (2.13). It can be found that

$$F(-p_n) = -2\int_0^1 [f_1(y)\, h(p_n, y) + f_2(y)\, t(p_n, y)]\, dy - H(p_n) \qquad (2.20)$$

The unknowns $\sigma_x(0, y)$, $\sigma_{xy}(0, y)$ for problem 4 enter into all the conditions (2.6), (2.8) — (2.11). Without determining them it is impossible to obtain either the solution of the problem or the damping conditions.

Hence, $\sigma_x(0, y)$, $\sigma_{xy}(0, y)$ should first be determined from conditions (2.6), (2.8), (2.9). (The method to determine them is given in [7] (*)) . Afterwards, we obtain the two damping conditions imposed on the boundary functions $f_1(y)$, $f_2(y)$ from (2.10) and (2.11).

3. Let us consider the symmetric strain of a half-strip. Then $a_1(p)$, $a_3(p)$ are zero in (1.9), while $y_1 = 0$, $y_2 = -1$ in (1.11). Determining $a_2(p)$, $a_4(p)$ from (1.12), we obtain

$$\begin{aligned} a_2(p) = \frac{1}{p\varphi(p)} \Big\{ &-\frac{1}{2\mu}(G_1(p) + \lambda u(0, 1))\left(\frac{\mu}{\lambda + \mu}\sin p - p\cos p\right) + \\ &\frac{1}{2\mu}(G_2(p) + \lambda v(0, 1))\left(\frac{\lambda + 2\mu}{\lambda + \mu}\cos p + p\sin p\right) - \\ &p b_1(p, 1)\left(\frac{\mu}{\lambda + \mu} + \cos^2 p\right) + p b_3(p, 1)\left(\frac{\mu(\lambda + 2\mu)}{(\lambda + \mu)^2} + p^2\right)\Big\} \end{aligned} \qquad (3.1)$$

$$a_4(p) = \frac{1}{p\varphi(p)} \left\{\frac{1}{2\mu}(G_1(p) + \lambda u(0, 1))\sin p - \right.$$

*) There is a misprint in formula (4.2) for $\Phi_0(y)$ in [7]. It should be considered that

$$\Phi_0(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$$

$$\frac{1}{2\mu}(G_2(p)+\lambda v(0,1))\cos p +$$
$$pb_1(p,1)+pb_3(p,1)\left(-\frac{\lambda+2\mu}{\lambda+\mu}+\cos^2 p\right)\Big\}$$
$$\varphi(p)=\sin p\cos p + p$$

The equation $\varphi(p)=0$ has a first order root at the origin and an infinite set of four complex first order roots $p_n, \bar{p}_n, -\bar{p}_n, -p_n$. In conformity with this, $U(p, y)$ has a second order pole at $p=0$ and a first order pole and complex roots of the equation $\varphi(p)=0$, and $V(p, y)$ is a simple pole in all the roots of the equation $\varphi(p)=0$.

Equating the residues $U(p, y)e^{px}$, $V(p, y)e^{px}$ relative to the poles p_n with positive real part to zero, we have the system of conditions

$$F(p_n)+H(p_n)=0 \quad (n=1,2,\ldots) \tag{3.2}$$

where

$$F(p_n)=\int_0^1[\sigma_x(0,y)h(p_n,y)+\sigma_{xy}(0,y)g(p_n,y)+$$
$$2\mu u(0,y)s(p_n,y)+2\mu v(0,y)t(p_n,y)]\,dy$$
$$H(p_n)=-\frac{\lambda+2\mu}{\lambda+\mu}\left[\sin p_n\int_0^1 g_1(x)e^{-p_nx}dx-\cos p_n\int_0^1 g_2(x)e^{-p_nx}dx\right]-$$
$$\int_0^1\left[h(p_n,y)\int_0^\infty r_x(x,y)e^{-p_nx}dx+g(p_n,y)\int_0^\infty r_y(x,y)e^{-p_nx}dx\right]dy$$
$$h(p,y)=py\sin py-\left(\frac{\mu}{\lambda+\mu}+\cos^2 p\right)\cos py$$
$$g(p,y)=-py\cos py+\left(\frac{\mu}{\lambda+\mu}+\sin^2 p\right)\sin py$$
$$s(p,y)=p(py\sin py-(1+\cos^2 p)\cos py)$$
$$t(p,y)=p(-py\cos py-\cos^2 p\sin py)$$

Equating the residues $U(p, y)e^{px}$, $V(p, y)e^{px}$ relative to the pole $p=0$ to zero, we obtain the two conditions

$$\int_0^1\sigma_x(0,y)\,dy=\alpha \tag{3.3}$$

$$\frac{\lambda}{2(\lambda+\mu)}\int_0^1\sigma_{xy}(0,y)y\,dy+2\mu\int_0^1 u(0,y)\,dy=\beta \tag{3.4}$$

Here

$$\alpha=\int_0^\infty\left[g_1(x)+\int_0^1 r_x(x,y)\,dy\right]dx$$

$$\beta=\frac{\lambda+2\mu}{2(\lambda+\mu)}\int_0^\infty\left[-xg_2(x)+\int_0^1\left(-xr_x(xy)+\frac{\lambda}{\lambda+2\mu}yr_y(x,y)\,dy\right)\right]dx$$

Let us turn to the construction of the damped solution for a half-strip. As in the case of the skew-symmetric strains, we represent $U(p, y)$, $V(p, y)$ as the sum of two terms, the first of which are transforms for the displacements in the absence of volume forces and stresses on the longitudinal boundaries, and the second contain transforms of the volume forces and stresses at $x = \pm 1$. We use the inversion theorem to find the originals corresponding to the first term. We use the convolution theorem, the inversion theorem, and the theorem on integration with respect to a parameter to construct the originals for the second terms. We consequently obtain

$$u(x, y) = \sum_{n=1}^{\infty} (u^*(-p_n, x, y) + u^*(-\bar{p}_n, x, y)) \tag{3.5}$$

$$v(x, y) = \sum_{n=1}^{\infty} (v^*(-p_n, x, y) + v^*(-\bar{p}_n, x, y))$$

$$u^*(-p_n, x, y) = -\frac{\lambda+\mu}{4\mu(\lambda+2\mu)} \frac{1}{p_n \cos^2 p_n} [F(-p_n) e^{-p_n x} + K(-p_n, x)] h(p_n, y)$$

$$v^*(-p_n, x, y) = -\frac{\lambda+\mu}{4\mu(\lambda+2\mu)} \frac{1}{p_n \cos^2 p_n} [F(-p_n) e^{-p_n x} + K(-p_n, x)] g(p_n, y)$$

$$K(-p_n, x) = \int_0^x \left\{ \frac{\lambda+2\mu}{\lambda+\mu} (\sin p_n g_1(\xi) + \cos p_n g_2(\xi)) - \int_0^1 [-h(p_n, y) r_x(\xi, y) + g(p_n, y) r_y(\xi, y)]\, dy \right\} e^{-p_n(x-\xi)} d\xi$$

Let us examine the question of the damping conditions for problems (1.1) — (1.4). We obtain the damping condition for problem 1 from (1.1) and (3.3)

$$\int_0^1 f_1(y)\, dy = \alpha$$

This condition is obtained from the static equilibrium conditions for a half-strip. Conditions (3.2),(3.4) are used to determine the unknowns $u(0, y)$, $v(0, y)$ in (3.5). (The method is indicated in [7]).

We have the damping condition

$$\frac{\lambda}{2(\lambda+\mu)} \int_0^1 f_2(y)\, y\, dy + \int_0^1 f_1(y)\, dy = \beta$$

for problem 2 from (1.2) and (3.4).

The system of conditions (3.2) permits elimination of the unknowns $v(0, y)$, $\sigma_x(0, y)$ in the expression for $F(-p_n)$ from the solution (3.5). We consequently obtain

$$F(-p_n) = -2\int_0^1 [f_2(y) g(p_n, y) + f_1(y) s(p_n, y)]\, dy - H(p_n)$$

For problem 3 we obtain the same damping condition from (1.3) and (3.3) as for problem 1. The system of conditions (3.2) permits elimination of the unknowns $u(0, y)$, $\sigma_{xy}(0, y)$ from the solution (3.5). In this case we have for $F(-p_n)$

$$F(-p_n) = 2\int_0^1 [f_1(y) h(p_n, y) + f_2(y) t(p_n, y)]\, dy + H(p_n)$$

For problem 4 the unknowns $\sigma_x(0, y)$, $\sigma_{xy}(0, y)$ enter into all the conditions (3.2)–(3.5) and into the solution (3.5). The system of conditions (3.3), (3.2) permits determination of the unknowns $\sigma_x(0, y)$, $\sigma_{xy}(0, y)$. From (3.4) we afterwards obtain the condition imposed on the function $f_1(y)$, which it is necessary and sufficient to satisfy in order for the damping solution to exist.

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